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PRECRITICAL EQUILIBRIUM OF A THIN SHALLOW SHELL OF REVOLUTION AND ITS STABILITY*

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A foundation is given for asymptotic expansions for the axisymmetric precritical equilibrium of a very thin elastic, strictly-convex, shell of revolution with fixed clamping of the edge, and for the derivation of an asymptotic estimate for the critical load for the loss of stability of this equilibrium for axisymmetric shell buckling. The question of the domain of applicability of the formulas for the asymptotic values of the upper critical loads /1,2/ obtained by the boundary layer method is discussed.

1. **On the formulation of the problem.** Considered is a system of equations of axisymmetric deformation of thin shallow elastic shells of revolution /3/ with the boundary conditions

$$\varepsilon^2 Av - \frac{1}{2}u^2 + \theta u = 0,\ \ \varepsilon^2 Au + uv - \theta v + \varphi(r) = 0,\ \ A(\ ) \equiv -r\frac{d}{dr}\frac{1}{r}\frac{d}{dr}r(\ ),\ \ \varphi(r) = \int_0^r q(t)\,t\,dt \tag{1.1}$$

$$\theta \leqslant -\alpha^2 r,\ \alpha = \text{const} > 0,\ \ \varepsilon^2 = ha^{-1}[12(1-\nu^2)]^{-1/2}$$

$$\left|\frac{v}{r}, \frac{u}{r}\right|_{r=0} < \infty,\ \ \left[\frac{dv}{dr} - \frac{\nu}{r}v\right]_{r=1} = 0,\ \ \left[\frac{du}{dr} + \left(\frac{\nu}{r} + \frac{k}{\varepsilon}\right)u\right]_{r=1} = 0,\ \ k > 0 \tag{1.2}$$

where k is the coefficient of elastic clamping of the edge in a fixed wall. For $k = 0$ and $k = \infty$, respectively, (1.2) corresponds to fixed hinged clamping and absolute clamping of the edge. All the quantities in (1.1) and (1.2) are dimensionless and associated with the dimensional relationships mentioned in /1/. No constraints are imposed on the sign of the transverse pressure $q(r)$.

Asymptotic expansions

$$v \sim v_\varepsilon \equiv \sum_{i=0}^n \varepsilon^i [v_i(r) + h_i(t)],\ \ u - u_\varepsilon \equiv \sum_{i=0}^n \varepsilon^i [u_i(r) + g_i(t)],\ \ t = (1-r)\varepsilon^{-1},\ v_0 = \varphi\theta^{-1},\ u_0 = 0 \tag{1.3}$$

are constructed according to /1/ for the equilibrium modes in the precritical stage. Here v_i, u_i are determined from the algebraic systems

$$\theta u_i - {}^1/_2 \sum_{k+j=i} u_k u_j + Av_{i-2} = 0,\ \ \sum_{k+j=i} u_k v_j - \theta v_i + Au_{i-2} = 0 \tag{1.4}$$

where $u_{-1} = v_{-1} \equiv 0$. The edge effect functions h_i, g_i are found successively from the equations with the boundary conditions

$$h_i'' + g_i(\eta_i g_0 - \theta_0) = th_{i-1}'' + h_{i-1}' + \sum_{k+j+2=i} t^k h_j - {}^1/_2 \sum_{m+n=i} g_m g_n + \sum_{l+p=i} t^l \theta_l g_p - \sum_{k+l+j=i} t^l u_{kl} g_j \equiv F_{i1},\ \ g_i'' - \eta_i(h_i g_0 + g_i h_0) + h_i\theta_0 - v_0(1) g_i = tg_{i-1}'' + g_{i-1}' + \sum_{k+j+2=i} t^k g_j + \sum_{m+n=i} h_m g_n + \sum_{m+l+p=i} u_{ml} t^l h_p - \sum_{l+p=i} \theta_l t^l h_p + \sum_{k+l+p=i} v_{kl} t^l g_p \equiv F_{i2} \tag{1.5}$$

$$m \neq 0,\ n \neq 0,\ p \neq i;\ \eta_0 = \frac{1}{2},\ \eta_i = 1\ (i \geqslant 1)$$

$$\{\theta_l, u_{ml}, v_{ml}\} = \frac{(-1)^l}{l!}\frac{d^l}{dr^l}\{\theta, u_m, v_m\}_{r=1},\ \ l = 0, 1, 2, \ldots,\ \ (\ )' = d(\ )/dt,\ \ \{h_i, g_i\}_{t=\varepsilon^{-1}\to\infty} = 0$$

$$h_i'(0) = \left[\frac{dv_{i-1}}{dr} - \nu v_{i-1}\right]_{r=1} - \nu h_{i-1}(0),\ \ h_{-1} = g_{-1} = 0,\ \ [g_i' - kg_i]_{t=0} = \left[\frac{du_{i-1}}{dr} + \nu u_{i-1} + ku_i\right]_{r=1} + \mu g_{i-1}(0) \tag{1.6}$$

For $i = 0$ the problem (1.5),(1.6) has the trivial solution $h_0 = g_0 \equiv 0$, from which no new solutions branch off for $\varphi_0 = \varphi(1) < 2\theta_0^2$ since the corresponding linearized problem has no eigenvalues. In this case, we have a system of linear differential equations with constant coefficients to determine h_i, g_i for $i \geqslant 1$. In particular, for $i = 1$ we find

$$h_1 = \frac{B}{2ab}\left[a\left(\frac{2a}{z_0} + \frac{Q}{2}\right)x + b\left(\frac{2a}{z_0} + \frac{Q}{2} - 2\right)y\right]\qquad g_1 = \frac{B}{b}\left[\left(1 - \frac{a}{z_0}\right)x + \frac{by}{z_0}\right],\ \ z_0 = 2a + k(-\theta_0)^{-1/2} \tag{1.7}$$

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$$x = e^{-a\tau}\sin b\tau,\quad y = e^{-a\tau}\cos b\tau,\quad \tau = \frac{1-r}{\varepsilon}(-\theta_0)^{1/2},\quad Q = \frac{2\varphi(1)}{\theta_0^2},\quad B = (-\theta_0)^{-1/2}\left[\frac{d}{dr}(\varphi\theta^{-1}) - \nu\varphi\theta^{-1}\right]_{r=1}$$

$$\theta_0 = \theta(1),\quad a = \left(\frac{4-Q}{8}\right)^{1/2},\quad b = \left(\frac{4+Q}{8}\right)^{1/2},\quad \varphi(1) < 2\theta_0^2$$

(The misprints in (2.5) and (2.9) from /1/ are eliminated in (1.5) and (1.7)). For $k=0$ and $k=\infty$, formulas (1.7) go over into the corresponding formulas (2.9) from /1/. It is easy to establish by induction that h_n, g_n consist of linear combinations of elements of the form

$$t^m \exp[-(j-m)(a \pm ib)\tau],\quad m = 0, 1, \ldots, n-1;\quad i = \sqrt{-1} \tag{1.8}$$

$$m < j = 1, 2, \ldots, n;\quad \varphi(1)/\theta_0^2 < 2(1-\delta),\quad \delta = o(1)$$

The asymptotic expansions (1.3) constructed are formal in nature and correspond to pre-critical equilibrium only under definite conditions. A foundation is given below for the asymptotic as $\varepsilon \to 0$. For simplicity in the exposition, only the limit cases $k=0$ and $k=\infty$ are considered, i.e., the boundary conditions 3) and 4) in (1.2) from /1/.

2. Foundation for the asymptotic.

There results from a sequential analysis of the problems (1.4)

Lemma 2.1. Let $M \geqslant n$, $\varphi(r) = \varphi_1(r^2)$, $\varphi(0) = 0$ and $z(r) = z_1(r^2)$, where the functions $\varphi_1(s), z_1(s)$ are, respectively, continuously differentiable $M+2$ and $M+3$ times. Then the solutions of the problems (1.4), the functions u_i, v_i, are twice continuously differentiable for $0 \leqslant r \leqslant 1$ and the following relationships hold

$$u_{2k+1} = v_{2k+1} = 0,\quad k = 0, 1, \ldots, [\tfrac{1}{2}(n-1)];\quad u_{2k} = O(r)$$

$$v_{2k} = O(r),\ Au_{2k} = O(r^2),\ Av_{2k} = O(r^2),\ k = 0, 1, \ldots, [n/2]$$

Let us introduce the notation

$$\varphi_n = \varphi(r)\theta^{-1}(r) + \varepsilon s_2(r,\varepsilon),\quad \psi_n = \varepsilon s_1(r,\varepsilon) \tag{2.1}$$

$$s_1(r,\varepsilon) = \sum_{i=1}^{n}\varepsilon^{i-1}u_i + \sum_{i=1}^{N_0}\varepsilon^{i-1}(g_i + \beta_i) + \varepsilon^n\gamma_1,\qquad s_2(r,\varepsilon) = \sum_{i=1}^{n}\varepsilon^{i-1}v_i + \sum_{i=1}^{n+1}\varepsilon^{i-1}(h_i + \alpha_i) + \varepsilon^n\gamma_2$$

Here α_i, β_i are infinitely differentiable monotonic functions of exponential order of smallness in ε, where $\beta_i(r) = -g_i(\varepsilon^{-1})$, $\alpha_i(r) = -h_i(\varepsilon^{-1})$ for $0 \leqslant r \leqslant 0.1$ and $\beta_i(r) = \alpha_i(r) \equiv 0$ for $0.2 \leqslant r \leqslant 1$, while the arbitrary sufficiently smooth functions $\gamma_1(r)$ and $\gamma_2(r)$ satisfy the relationships

$$|\gamma_1/r,\ \gamma_2/r|_{r=0} < \infty,\ \left[\frac{d\gamma_2}{dr} - \nu\gamma_2\right]_{r=1} = \nu h_{n+1}(0),\qquad \left[\frac{d\gamma_1}{dr} + \nu\gamma_1\right]_{r=1} = -\nu g_{n+1}(0),\quad N_0 = n+1,\quad k=0$$

If $k=\infty$, the $N_0 = n$ and $\gamma_1 \equiv 0$. The functions $(\varepsilon^i\beta_i, \varepsilon^i\alpha_i)$ and (γ_1, γ_2), respectively, cancel the residuals in satisfying the boundary conditions (1.2) for $r=0$ and $r=1$ for the expansion $(u_\varepsilon, v_\varepsilon)$. Consequently, φ_n, ψ_n satisfy all the boundary conditions of the problem exactly and the following estimates hold

$$\max\left|\frac{d^\varkappa z}{dr^\varkappa}\right| < m_1\varepsilon^{n+1},\quad \varkappa = 0, 1, 2 \tag{2.2}$$

where $z = u_\varepsilon - \psi_n$ or $z = v_\varepsilon - \varphi_n$. Here, as everywhere in Sect.2, m_i, c_i are certain positive constants independent of r and ε; the maximum is always taken for $0 \leqslant r \leqslant 1$.

Moreover, by applying Lemma 2.1, the relationships

$$\max|r^{-1}(g_i + \beta_i)| \leqslant \max\left|\frac{dg_i}{dr}\right| \tag{2.3}$$

and analogous estimates for h_i, we have the inequalities

$$|\varphi_n - \varphi\theta^{-1}| \leqslant m_2 r\varepsilon,\quad |\psi_n| \leqslant m_2 r\varepsilon,\quad |\varphi\theta^{-1}| \leqslant m_0 r \tag{2.4}$$

Lemma 2.2. Let the conditions of Lemma 2.1 be satisfied, $\varepsilon \to 0$, and let δ_2 be an arbitrarily small positive number independent of ε, $\delta_2 = o(1)$. Then for $Q < 4 - \delta_2^2$, the following estimates hold for (φ_n, ψ_n):

$$F_1(r,\varepsilon) \equiv \varepsilon^2 A\varphi_n - {}^1/_2\psi_n^2 + \theta\psi_n = O(r\varepsilon^{n+1}),\qquad F_2(r,\varepsilon) \equiv \varepsilon^2 A\psi_n + \varphi_n\psi_n - \theta\varphi_n + \varphi(r) = O(r\varepsilon^{n+1}) \tag{2.5}$$

$$(|F_1(r,\varepsilon)| \leqslant m_3 r\varepsilon^{n+1},\ |F_2(r,\varepsilon)| \leqslant m_3 r\varepsilon^{n+1})$$

Proof. We substitute (2.1) into (1.1) and we obtain by using (1.4):

$$F_1(r,\varepsilon) = \sum_{i=1}^{n+1}\varepsilon^{i+2}Ah_i^\circ - \sum_{m+k=1}^{n+N_0}\varepsilon^{m+k}u_m g_k^\circ - {}^1/_2\sum_{j+k=2}^{2N_0}\varepsilon^{j+k}g_j^\circ g_k^\circ + \tag{2.6}$$

$$\theta \sum_{k=1}^{N_0} \varepsilon^k g_k^\circ + \varepsilon^{n+3} A\gamma_2 + \varepsilon^{n+1}\gamma_1 \left[- \sum_{m=1}^{n} \varepsilon^m u_m - \sum_{k=1}^{n} \varepsilon^k g_k^\circ + \theta - {}^1/_2 \varepsilon^{n+1}\gamma_1 \right]$$

$$g_k^\circ = g_k + \beta_k, \quad h_k^\circ = h_k + \alpha_k, \quad 1 \leqslant m \leqslant n, \quad 1 \leqslant j, \quad k \leqslant N_0$$

The expression for $F_2(r, \varepsilon)$ is analogous in form. We consider the right side of (2.6) for $r \in [0, 0.25)$. We note that for $y_i = g_i^\circ$ or $y_i = h_i^\circ$ the following relationships hold

$$\lim_{r\to 0} \frac{Ay_i}{r} = -\frac{3}{2} \frac{d^2 y_i}{dr^2}\bigg|_{r=0}, \quad \left|\frac{Ay_i}{r}\right| \leqslant \frac{5}{2} \max \left|\frac{d^2 y_i}{dr^2}\right| \tag{2.7}$$

Now, by using (2.3) and (2.7) and taking into account that h_i, g_i together with their derivatives tend more rapidly to zero than any power of ε, on the segment $[0, 0.25)$ of the boundary layer for $Q < 4 - \delta_2^2$, we establish (2.5). For $r \in [0.25, 1]$, the estimate (2.5) results from (2.6) after going over to the variable t by using (1.5), (1.6), and estimates for the elements of the form (1.8).

Theorem 2.1. Let the conditions of Lemma 2.1 be satisfied for $M > 7$ $(M = N + 4)$, let $\varepsilon \to 0$ and let δ be an arbitrarily small positive number $(\delta = o(1))$ independent of ε, and

$$\max [\varphi(r)\, \theta^{-2}(r)] \leqslant 2(1 - \delta) \tag{2.8}$$

Then there is an ε_1 such that for $0 < \varepsilon < \varepsilon_1$ and a sufficiently small neighborhood of the asymptotic expansions $(v_\varepsilon, u_\varepsilon)$ from (1.3), there exists a unique solution (v, u) for each of the problems (1.1), (1.2), where the estimates

$$\max |v - v_\varepsilon| \leqslant m\varepsilon^{n+1}, \ \max |u - u_\varepsilon| \leqslant m\varepsilon^{n+1} \tag{2.9}$$

are valid for $n = 0, 1, \ldots, N$. Moreover, we have for $n = N + j$ $(j = 1, 2, 3, 4)$

$$\max |v - v_\varepsilon| \leqslant m\varepsilon^{N+1}, \ \max |u - u_\varepsilon| \leqslant m\varepsilon^{N+1}$$

Proof. We apply the method of giving a foundation to the asymptotic which has been developed in /4,5/. We will consider the problem (1.1) and (1.2) as an operator equation

$$\mathbf{P}(\mathbf{V}) = 0, \quad \mathbf{V} \equiv (v, u) \tag{2.10}$$

Here $\mathbf{V}$ is the solution and the operator P is determined by the left side of the system (1.1) and acts from the sapce X, the closure of the set of smooth vector-functions $\mathbf{V} \equiv (v, u)$ satisfying the boundary conditions (1.1) and (1.2) in the norm

$$\|\mathbf{V}\|_X^2 = \int_0^1 [(Av)^2 + (Au)^2]\, dr$$

to the space Y which is the vector-function $\mathbf{V}$ with the finite norm

$$\|\mathbf{V}\|_Y^2 = \int_0^1 (v^2 + u^2)\, dr = \|v\|^2 + \|u\|^2$$

According to /4,5/, to give a foundation to the asymptotic it must be proved that as $\varepsilon \to 0$ the following inequality is satisfied

$$\|\mathbf{P}(V_\varepsilon)\|_Y \|[P'_{\mathbf{V}_\varepsilon}]^{-1}\|_{(Y\to X)}^2 \|\mathbf{P}''\|_{(X\to(X\to Y))} < {}^1/_2 \tag{2.11}$$

where $\mathbf{V}_\varepsilon \equiv (\varphi_n, \psi_n)$ are the asymptotic expansions, $\mathbf{P}'_{\mathbf{V}_\varepsilon}$ is the Fréchet derivative on the element V_ε and $\mathbf{P}''$ is the second derivative of the operator $\mathbf{P}$.

Lemma 2.3. Let the conditions of Lemma 2.2 be satisfied. Then the following estimates hold

$$\|\mathbf{P}(\mathbf{V}_\varepsilon)\|_Y \leqslant c_1\varepsilon^{n+1}, \quad \|\mathbf{P}''\| \leqslant c_3, \quad \mathbf{V}_\varepsilon \equiv (\varphi_n, \psi_n) \tag{2.12}$$

Lemma 2.4. Let the conditions of Theorem 2.1 be satisfied. Then the following estimate holds

$$\|[\mathbf{P}'_{\mathbf{V}_\varepsilon}]^{-1}\|_{(Y\to X)} \leqslant c_2\varepsilon^{-4} \tag{2.13}$$

Proof. Let us consider the system of equations

$$\mathbf{P}'_{\mathbf{V}_\varepsilon}(\mathbf{V}) = f, \quad \mathbf{f} = (f_1, f_2), \quad \mathbf{V} \equiv (v, u), \quad \mathbf{P}'_{\mathbf{V}_\varepsilon}(\mathbf{V}) = \left\{\varepsilon^2 Av - \varepsilon s_1 u + \theta u, \ \varepsilon^2 Au + \frac{\varphi u}{\theta} + \varepsilon s_2 u + \varepsilon s_1 v - \theta v\right\} \tag{2.14}$$

where $\mathbf{P}_{\mathbf{V}_\varepsilon}$ is the Fréchet derivative on the element $\mathbf{V}_\varepsilon$.

For $k = \infty$ we multiply the first equation in (2.14) by $v - 2u + \delta u$ and the second by $u + \varepsilon v$. Combining and integrating by parts and taking the boundary conditions into account, we find

$$\varepsilon^2 I_1 + (2 - \delta) I_2 + \varepsilon I_3 + \int_0^1 \varphi\theta^{-1} u^2 dr - \varepsilon^2 \nu v^2(1) + I_4 + I_5 + I_6 - \varepsilon^2 v(1) u'(1) = \int_0^1 [f_1(v - 2u + \delta u) + f_2(u + \varepsilon v)]\, dr \tag{2.15}$$

$$I_1 = \int_0^1 \left(rv'^2 + ru'^2 + \frac{v^2}{r} + \frac{u^2}{r}\right) dr, \quad I_2 = -\int_0^1 \theta u^2 dr \geqslant 0, \quad I_3 = -\int_0^1 \theta v^2 dr \geqslant 0, \quad I_4 = \varepsilon \int_0^1 \varphi\theta^{-1} uv dr$$

$$I_5 = \varepsilon \int_0^1 [s_2 u^2 + (2-\delta) s_1 u^2 + \varepsilon s_2 uv + \varepsilon s_1 v^2]\, dr, \qquad I_6 = \varepsilon^2 (\delta + \varepsilon - 2) \int_0^1 \left(rv'u' + \frac{uv}{r}\right) dr$$

Let us note the following inequalities

$$\varepsilon^2 v^2(1) \leqslant \varepsilon^{1/2} m_1 I_1 + \varepsilon^{3/2} m_2 I_3, \quad |I_4| \leqslant m_3 (\varepsilon^{1/2} I_2 + \varepsilon^{3/2} I_3), \qquad |I_5| \leqslant m_4 \varepsilon (I_2 + \varepsilon I_3) \tag{2.16}$$

$$|I_6| \leqslant \varepsilon^2 (1 - {}^1/_4 \delta) I_1, \quad m_1 = \theta_0^{-2} \max(\theta^2 r^{-1})$$
$$m_2 = \theta_0^{-2} \max(|\theta| + 2\varepsilon^{1/2} |\theta'|), \quad m_3 = {}^1/_2 \max |\varphi\theta^{-2}|, \qquad m_4 = 2 \max [\theta^{-1}(|s_1| + |s_2|)], \quad \varepsilon < {}^1/_2\, \delta$$

In order to estimate the component $v(1)u'(1)$, we multiply the socond equation of (2.14) by ru', integrate between 0 and 1 and apply the Cauchy–Buniakovskii inequality. We consequently obtain

$$|\varepsilon^3 u'(1) v(1)| \leqslant \varepsilon^2 m_7 v^2(1) + {}^1/_4 \varepsilon^{3/2} (3\varepsilon I_1 + I_2 + I_3 + \| r^{1/2} f_2 \|^2), \quad m_7 = {}^1/_2 + m_5^2$$
$$m_5 = \max\left(\left|\frac{r\varphi^2}{\theta^3}\right|^{1/4} + \varepsilon^{1/2} \left|\frac{r s_2^2}{\theta}\right|, \quad |r\theta|^{1/4} + \varepsilon^{1/2} \left|\frac{r s_1^2}{\theta}\right|^{1/4}\right) \tag{2.17}$$

Finally, using (2.16) and (2.17), we have from (2.15)

$$ {}^1/_8 \varepsilon^2 \delta I_1 + 2(1-\delta) I_2 + \varepsilon (1-\delta) I_3 + \int_0^1 \varphi\theta^{-1} u^2 dr \leqslant \|f_1\| \, \|v - 2u + \delta u\| + \|f_2\| \, \|u + \varepsilon v\| + {}^1/_4 \varepsilon^{3/2} \|r^{1/2} f_2\|^2 \tag{2.18}$$

Hence ε satisfies the condition

$$\varepsilon^{1/2} (m_2 + m_3 + m_4 \varepsilon^{1/2} + m_2 m_7 + \varepsilon^2 m_1 + \varepsilon^2 m_1 m_7) < \delta$$

Applying (2.8), we obtain an estimate from (2.18)

$$\max |u| + \max |v| \leqslant m_8 \varepsilon^{-2} \|\mathbf{f}\|_Y \tag{2.19}$$

Now, (2.13) is derived from (2.14) analogously to /5/ by using (2.19).

In the $k = 0$ case, the estimate (2.13) was obtained in /1/ for the particular case of $\theta = -\lambda r$, $\varphi(r) = {}^1/_2 q r^2$. In the general case, we multiply the first equation of the system (2.14) by $\delta_1 v - u$, and the second by $\delta_1 u + v$, where δ_1 is some small positive number. Integrating between 0 and 1, and adding, we obtain

$$\delta_1 \varepsilon^2 [I_1 + \nu u^2(1) - \nu v^2(1)] + 2\nu\varepsilon^2 u(1) v(1) + I_2 + I_3 + \delta_1 \int_0^1 \varphi\theta^{-1} u^2 dr + \int_0^1 \varphi\theta^{-1} uv dr + I_7 = \int_0^1 [(\delta_1 f_1 + f_2) v + (\delta_1 f_2 - f_1) u]\, dr \tag{2.20}$$

$$I_7 = \varepsilon \int_0^1 (\delta_1 s_2 u^2 + \delta_1 s_2 uv + s_1 v^2 + s_1 u^2)\, dr$$

(The misprint in (5.7) from /1/ is eliminated here, where $\delta_1 s_2 uv + s_1 u^2$ should replace $s_2 uv + s_1 wv$ in an integrand of the form I_7). Applying the estimate

$$|I_7| \leqslant m_9 \varepsilon (I_2 + I_3), \qquad m_9 = \max [(|s_1| + 1.5\ \delta_1 |s_2|)\, \theta^{-1}]$$

and the first inequality in (2.16), we deduce from (2.20)

$$0.5\, \delta_1 \varepsilon^2 I_1 + (1 - 4\ \delta_1^2)(I_2 + I_3) - (0.5 + \delta_1)(I_2 + I_3) \times \max |\varphi\theta^{-2}| \leqslant (\|f_1\| + \|f_2\|)(\|v\| + \|u\|)$$

where ε satisfies the conditions

$$4 m_1 \varepsilon^{1/2} < \delta_1, \quad (2\nu + \delta_1) m_2 \varepsilon^{3/2} < 4\delta_1^2$$

Setting $\delta = 2\delta_1$, and taking account of (2.8), we hence find (2.19) and the estimate (2.13) analogously to /5/.

By using (2.12) and (2.13) we confirm that all the conditions of Theorem 3.2 from /5/, and particularly, conditions (2.11) are satisfied for $k = M$, $m = 4$, if $M > 7$ and ε is sufficiently small $(0 < \varepsilon < \varepsilon_1)$. Hence, the problem (2.10) has a solution for which the estimate $\|\mathbf{V} - \mathbf{V}_\varepsilon\| \leqslant m\varepsilon^{M-3}$ is valid. Now by applying the triangle inequality, we obtain the estimate (2.9) analogously to /5/.

3. Asymptotic estimates for the upper critical load. Lower estimates for the asymptotic value of the upper critical load result from Theorem 2.1 for axisymmetric snapping of shells with fixed clamping of the edge. For simplicity, we assume that the load functions depend on one parameter ρ, i.e., $\varphi(r, \rho)$ and $\varphi(r, 0) = 0$.

Theorem 3.1. Let the conditions of Theorem 2.1 be satisfied, let $\varepsilon \to 0$ and let δ be an arbitrarily small positive number $(\delta = o(1))$ independent of ε. Then the upper critical value ρ^* /1/ for the problem (1.1), (1.2) satisfies the inequality

$$\max\,[\varphi(r,\rho^*)\,\theta^{-2}(r)] > 2\,(1-\delta)$$

The theorem follows directly from the fact that a unique solution exists in the neighborhood of V_ε under the condition (2.8).

After the work of A. V. Pogorelov, it is well known that sufficiently thin, strictly-convex, shells can buckle with the formation of local dents far from the edge ("principle B " /6/, and also see /7—9/). A kind of buckling which corresponds to local buckling of the shell near the edge is considered in the author's papers /1,2/. Hence, the critical values mentioned in /1,2/, should be considered as asymptotic values of the upper critical loads of local buckling near the edge under the assumption that the number of azimuthal waves does not increase too rapidly as $\varepsilon \to 0$. In the general case these loads only yield an asymptotic upper estimate for the upper critical load, which is defined as the lowest branch point. When the local dents are not formed successfully, the upper critical load is not only estimated but is determined exactly by the critical buckling load in the edge effect zone. This case holds for the axisymmetric deformation of a spherical shell under uniform external pressure for a moving hinged support and sliding clamping of the edge, as well as in the case of fixed clamping of the edge /9/, when local dent formation occurs with the buckling of any part within the shell and on the edge for asymptotically coincident values of the pressure. Therefore, (4.4) in /1/ should be used when the function $\varphi\theta^{-2}$ has a maximum for $r = 1$.

In the general case, when the maximum of $\varphi\theta^{-2}$ is reached within the domain at a point r_*, and the dent is not small ($r \geqslant r_*$, $r_*\varepsilon^{-1} \gg 1$, local buckling starts far from the shell pole), the asymptotic formula

$$\sup_r\,(\varphi(r,\rho^*)\,\theta^{-2}(r)) = 2 \tag{3.1}$$

is formally obtained in /9/ for the upper critical load for the problem (1.1), (1.2).

In the case of the smoothness of $\varphi\theta^{-2}$, Theorem 3.1 assures that this formula yields an exact lower estimate for ρ^* when the mentioned maximum is reached for $0 \leqslant r \leqslant 1$.

Moreover, let the subscripts $j = 1$—4 correspond to the boundary conditions 1) —4) in (1.2) from /1/. Then for the boundary conditions 1) and 2) the upper critical load Ω_j^* (as the lowest branchpoint as $\varepsilon \to 0$) is determined by the formulas

$$\Omega_j^* = \min(\rho^*, \rho_j^*), \quad j = 1, 2 \tag{3.2}$$

where ρ_1^* and ρ_2^* are determined from the relationships /1/

$$\varphi(1,\rho_1^*)\,\theta^{-2}(1) = 0.3965, \quad \varphi(1,\rho_2^*)\,\theta^{-2}(1) = 0.8835$$

Let us present examples of the numerical computations for spherical shells ($\theta = -r$) subjected to loads varying along the radius. Let $q = 4\rho\,(1-r^2)$. For $b = 30, 75, 150, 250$ in cases 1)—4), we have for the critical values, respectively, $\rho_1^c = 0.366, 0.384, 0.390, 0.392$; $\rho_2^c = 0.780, 0.843, 0.864, 0.872$; $\rho_{3,4}^c = 1.052, 1.010, 1.005, 1.003$. These results agree well with (3.1) and (3.2) which yield the asymptotic values $\rho_1^c(\infty) = 0.396$, $\rho_2^c(\infty) = 0.883$, $\rho_{3,4}^c(\infty) = 1.00$. The numerical results for ρ_4^c are given in /10/ for $b \leqslant 12$. Let $q = 4\rho r^2$. Then the numerical computations for the critical values ρ_j^c yield up to a 5% discrepancy from the asymptotic values for $j = 1,2,3,4$, respectively, when $b \geqslant 20, 40, 100, 400$. Here, for fixed clamping of the edge $\rho_{3,4}^c(b)$ slowly emerge on the asymptotic.

Both symmetric and nonsymmetric (with a different number of azimuthal waves) buckling modes can correspond to the upper critical load when nonaxisymmetric deformations are taken into account /10—13/. The result depends in a sufficiently complex manner on the shell shape and the pressure distribution (*). As an example of nonaxisymmetric buckling, we mention spherical shells with one of the following pressure distributions $4\rho\sin(\pi r/2)$, $4\rho r^m$, $4\rho\,(1+r^m)$, $m = 2, 4$. In these cases the numerical and asymptotic computations by using (1.3) and (1.7) show that the critical buckling loads in the nonaxisymmetric modes ρ_j^H are less than the corresponding axisymmetric buckling loads ρ_j^c. Meanwhile, both axisymmetric and nonaxisymmetric buckling modes are possible under the external pressure $4\rho\,(1-\alpha r^2)$, depending on the parameter α, as follows from the results presented. For $0.3 \leqslant \alpha \leqslant 1.0$ buckling occurs in axisymmetric modes, where $\rho_{3,4}^c(\infty) = 1.0$. Here $\sigma_i^* = \lim n^2/b^2$ as $n \to \infty$, $b \to \infty$.

Table.1

α	$\rho_{3,4}^c(\infty)$	$\rho_3^H(\infty)$	σ_3^*	$\rho_4^H(\infty)$	σ_4^*
0.1	1.0	0.793	0.613	0.912	0.705
0.2	1.0	0.897	0.568	1.036	0.644
0.3	1.0	1.025	0.519	1.170	0.580

4. Shallow spherical shell under uniform external pressure.

As is known, the critical pressures of uniformly loaded spherical shells are defined by the formulas

$$p_i(b) = \frac{2Eq_i(b)}{\sqrt{3(1-\nu^2)}}\left(\frac{h}{R}\right)^2, \quad b^2 = \sqrt{12(1-\nu^2)}\,\frac{a^2}{hR} \tag{4.1}$$

*) See /21/ as well as the paper by Bermus, I. M. and L. S. Srubshchik, "Application of numerical and asymptotic methods to compute the upper critical loads of elastic spherical shells". Rostov Univ., 1979. VINITI Dep. No. 2378-79

where h is the shell thickness, R is the sphere radius, a is the radius of the reference outline, ν is the Poisson ratio, and E is Young's modulus. Values of $q_i(b)$ have been calculated by many authors for axisymmetric buckling and are found in /10/ for $b \leqslant 60$ when $i = 1$ and $b \leqslant 42$ when $i = 2, 3, 4$. For large b the critical pressures emerge on the asymptotic values /6,1/

$$q_1(\infty) = 0.198,\quad q_2(\infty) = 0.442,\quad q_3(\infty) = q_4(\infty) = 1.0 \tag{4.2}$$

Numerical computations of the upper critical loads were performed for $b \leqslant 300$ on the BESM-6 computer to determine the limits of applicability of the asymptotic.

The numerical program was compiled by means of a modified algorithm from /13/ and was checked for $b \leqslant 42$ by a comparison with the results in /10/. The value of q at which a change in sign of the function

$$D(q) = \frac{\det B_2}{|\det B_2|} \prod_{i=2}^{N} \frac{\det(B_i + A_i Q_{i-1})}{|\det(B_i + A_i Q_{i-1})|} \det(G + HQ_N - GQ_{N-1}Q_N)$$

occurred in the interval $(q_i, q_i + \Delta q)$ where $\Delta q < 10^{-4}$, was taken as the critical value q_i, where the notation of /13/ is conserved. Part of the values $q_n^\circ = 10^4 \times q_n(b)$ found for $\nu = 0.33$ are shown below

Table.2

b	60	100	150	250
$\overset{\circ}{q}_1$	1945	1958	1965	1970
$\overset{\circ}{q}_2$	4343	4376	4393	4405
$\overset{\circ}{q}_3$	9722	9845	9900	9942
$\overset{\circ}{q}_4$	9829	9996	10000	9985

For mobile clamping of the edge $(i = 1,2)$ the discrepancy between the numerical and asymptotic values /1/ does not exceed 5% for $b \geqslant 20$ and 1% for $b \geqslant 100$. For fixed hinge clamping, this discrepancy is not more than 4% for $b \geqslant 45$ and 1% for $b \geqslant 150$, and in the case of complete clamping of the edge /6/ is not more than 5% for $b \geqslant 30$ and 1% for $b \geqslant 85$.

Let us note that axisymmetric buckling is, as a rule, associated with the snapping phenomenon in this problem, i.e., the critical values are limit points. The merger points $q_4(b)$ for $b_1 = 8.348$, $b_2 = 14.987$, detected in /14/, are the exception. Investigation using the alignment method showed that $q_4(b_1)$ and $q_4(b_2)$ are bifurcation points. (An analogous investigation was not performed for $b \geqslant 15$).

These results from Theorems 2.1 and 3.1 for uniformly loaded spherical shells

Theorem 4.1. Let $\varepsilon \to 0$ and $\theta = -\lambda r$, $\lambda = a/R$, $\varphi(r) = \frac{1}{2} qr^2$. Then for an arbitrarily small $\delta > 0$ $(\delta = o(1))$ there is a value of ε_1 such that for $0 < \varepsilon < \varepsilon_1$ the upper critical pressure q^* in the problem (1.1), (1.2) satisfies the inequality $q^* \geqslant 4\lambda^2 - \delta$. Hence for all $q < 4\lambda^2 - \delta$ in a sufficiently small neighborhood $v_\varepsilon, u_\varepsilon$ of (1.3) only one solution exists and for $n = 0, 1, 2, \ldots$ the following estimates are valid

$$\max|v - v_\varepsilon| \leqslant m\varepsilon^{n+1},\ \max|u - u_\varepsilon| \leqslant m\varepsilon^{n+1} \tag{4.3}$$

The inaccuracy the author made in formulating this theorem in case 3) in (1.2) (see /1/, p. 712) is eliminated here. Instead of $\delta = O(\varepsilon)$ in /1/ there should be $\delta = o(1)$.

Let us note that (4.3) improves the analogous estimate (5.2) in /1/ in case $k = 0$. For $k > 0$ in (1.2), the theorem is formulated for the first time. The author does not know of a rigorous proof for the upper estimate for $q_3(\infty)$ and $q_4(\infty)$.

By using the Marguerre—Vlasov equations in the problem of a uniformly loaded spherical shell under hard framing of the edge, Huang /11/ evaluated the critical pressures $p_H(b)$ at which the axisymmetric equilibrium mode can be buckled into the nonaxisymmetric mode. In particular, as $b \to \infty$, $n \to \infty$, the appropriate asymptotic value was found $p_H(\infty) = 0.810\ p_4(\infty)$, $n^2 b^{-2} \equiv \sigma \to 0.728$, where n is the number of the eigenfunction harmonic corresponding to the value $p_H(b)$ (here the corrected value of $p_H(\infty)$ is presented /12/). Let us also note that the values of $p_H(\infty)$ are given in /21/ as a function of k.

The experimental data /15/ turned out to be close to the Huang critical loads, however, they do not yield favorable results for nonaxisymmetric theory. In fact, according to computations /11/, buckling for $b \geqslant 5.5$ should be nonaxisymmetric in nature, and the number n (waves in the azimuthal direction) should grow together with b, and in experiments /15/ six out of nine shells had an axisymmetric mode after the experiment for $b \geqslant 5.5$, and the remaining three with $n = 1$ could be inelastic, as is shown in /16/, under loads close to the critical values. Moreover, all the shells tested in /15/ did not satisfy the shallowness criterion.

Later more precise experiments /16—20/ did not confirm the Huang theory, but displayed good agreement with existing symmetric theory. In particular, it the experimentally observed phenomenon of an axisymmetric edge effect is described in /20/.

The clarification of the reasons for such a disagreement between the experimental data and the results of the theory requires additional investigations.

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